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Department of Mathematics and Statistics
American University of Sharjah
Final Exam - Spring 2017
MTH 221 – Linear Algebra

Date: Monday, May 15, 2017 **Time:** 2:00 pm - 4:00 pm

Student Name	Student ID Number
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Instructor Name	Class Time
Dr. Ayman Badawi	12pm

1. ***Do not open*** *this exam until you are told to begin.*
2. *No questions are allowed during the examination.*
3. *This exam has **11** questions.*
4. *Do not separate the pages of the exam.*
5. *Scientific calculators are allowed.*
6. *Turn off all cell phones and remove all headphones.*
7. *Take off your cap.*
8. *No communication of any kind is allowed during the examination*
9. *If you are found wearing a smart watch or holding a mobile phone at any point during the exam then it will be considered an academic violation and will be reported to the dean's office.*

Student signature: U Moz Iqbal

Name [illegible], ID [illegible]
IQBAL

Final Exam, MTH 111, Fall 2016

Ayman Badawi

$\boxed{Q1}$ (10 pts) Find the values of h and k so that the following system

$$\begin{aligned} x - y + kz &= 0 \\ hy + kz &= 1 \\ y + kz &= 0 \end{aligned}$$

$1\begin{pmatrix} h & k \\ 1 & k \end{pmatrix} = hk - k = 0.$

$k(h-k) = 0.$

$k = 0 \quad h = k.$

a) has no solution,

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & k & 0 \\ 0 & h & k & 1 \\ 0 & 1 & k & 0 \end{array}\right] \begin{array}{l} -R_2+R_3 \to R_3 \\ -R_2+R_1 \to R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -h-1 & 0 & -1 \\ 0 & h & k & 1 \\ 0 & -h+1 & 0 & -1 \end{array}\right]$$

For no solution either.

$\{ h = 1 \text{ and } k \in \mathbb{R} \}$ or $\{ \cancel{h=0} \text{ and } k = 0 \}$.

$h \in \mathbb{R}$ and

$$\begin{vmatrix} 1 & -1 & k \\ 0 & h & k \\ 0 & 1 & k \end{vmatrix} = \begin{vmatrix} h & k \\ 1 & k \end{vmatrix} = hk - k = 0.$$

$k = 0$ or $k(h-1) = 0$

$h = 1.$

b) has infinitely many solutions,

There exist no h or k where solution is infinite.

c) has a unique solution.

$k \neq 0$ and $h \neq \mathbb{R}$. Needs k to be leader.

Q2 (5 pts) Determine if the polynomials $\{x^2+1,\ x^2-x+1,\ x-2\}$ are dependent or independent.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3 leaders, 3 rows independant

Q3 (5 pts) Calculate A^{-1} and show your work if $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -4 & -1 & -2 & 0 & 1 \end{array}\right]$$

$-2R_2+R_1\to R_1$

$4R_2+R_3\to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 4 & 1 \end{array}\right] \xrightarrow{R_3+R_1\to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 4 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -4 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & -4 & -1 \end{bmatrix}$$

Q4 **(10 pts)**

a) Suppose that A and B are some 3×3 matrices. If A is invertible and $BAB = 2A^T$, then what is $|B|$?

A invertible $(A^T)^{-1} = (A^{-1})^T$

$BAB(A^T)^{-1} = 2A^T(A^T)^{-1}$

$BAB(A^T)^{-1} = 2I_3$.

$|2I_3| = 2^3|I_3| = 8$.

$|B| \times |A| \times |B| \times |A^T|^{-1} = 8$.

$|A^T| = |A|$ and $|A| \times |A^{-1}| = 1$

$|B|^2 = 8$.

$|B| = \pm\sqrt{8}$.

b) Suppose that A and B are some invertible matrices. Find the matrix X (i.e. write X in terms of A, B and C) if

$$A^2X^TB = C.$$

$A^2X^TB = C$

A and B are invertible

$A^2X^T\cancel{BB^{-1}} = CB^{-1}$

$[A^2X^T]^T = [CB^{-1}]^T$

$X[A^2]^T = (B^{-1})^TC^T$

$[A^{-1}]^T = [A^T]^{-1}$

$[A^T][\cancel{A^T}][\cancel{A^T}]^{-1} = (B^{-1})^TC^T[A^T]^{-1}$

$X\cancel{A^T \times (A^T)^{-1}} = (B^{-1})^TC^T[A^T]^{-1}[A^T]^{-1}$

$X = (B^{-1})^T(C^T)([A^{-1}]^T)^2$

Q5 **(12 pts)** For each of the following, determine whether W is a subspace or not. If W is a subspace, then convince me that it is indeed a subspace and find its independent-number (i.e., $dim(W)$). If W is not a subspace give me an example showing that one of the axioms fails.

a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times 2} \mid ad = 0 \right\}$

either $a=0$ or $d=0$.

Let V_1 and $V_2 \in W$

$V_1 = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

$V_1 + V_2 = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \Rightarrow 1 \times 1 = 1 \neq \underline{\underline{0}}$

AXIOM 2 FAILS.

b) $W = \{(a+b, 2a+2b, -a-b) \in R^3 \mid a, b \in R\}$

$W = \{a(1,2,-1) + b(1,2,-1) / a \text{ and } b \in \mathbb{R}\}$.

$W = \text{Span}\{\underbrace{(1,2,-1)}_{Q_1}, \underbrace{(1,2,-1)}_{Q_2}\}$ written as a span / subspace

$Q_1 = Q_2$.

$W = \text{Span}\{(1,2,-1)\} \Rightarrow \dim(W) = \underline{\underline{1}}$.

c) $W = \{f(x) \in P_3 | f(1) = 0\}$.

A polynomial in P_3 is $ax^2 + bx + c = 0$.

$W = \{ax^2 + bx + c / f(1) = 0\}$

$W = \{ax^2 + bx + c / a(1) + b(1) + c = 0\}$

$W = \{ax^2 + bx + c / a + b + c = 0\}$.

$W = \{ax^2 + bx + c ; a = -c - b / c, b \in \mathbb{R}\}$.

$W = \{(-c-b)x^2 + bx + c / c, b \in \mathbb{R}\}$

$W = \{+c(-x^2+1) + b(-x^2+x) / c, b \in \mathbb{R}\}$

$W = \text{span}\{-x^2+1, -x^2+x\}$.

written as a span subspace!!

$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$-R_1 + R_2 \rightarrow R_2$

$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow 2$ leaders

$\dim W = \underline{\underline{2}}$

Q6 (10 pts) Let $Q_1 = (-2, 1, 3), Q_2 = (4, -2, -6), Q_3 = (2, 0, 5)$, and let $W = span\{Q_1, Q_2, Q_3\}$.

a) Can we write the point $(1, -2, 5)$ as a linear combination of the points Q_1, Q_2 and Q_3? SHOW THE WORK.

$$\left[\begin{array}{ccc|c} -2 & 4 & 2 & 1 \\ 1 & -2 & 0 & -2 \\ 3 & -6 & 5 & 5 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ -2 & 4 & 2 & 1 \\ 3 & -6 & 5 & 5 \end{array}\right] \underset{-3R_1+R_3 \to R_3}{\overset{2R_1+R_2 \to R_2}{\sim}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 5 & 11 \end{array}\right] \quad \left.\begin{array}{l} 2x_3 = -3 \\ 5x_3 = 11 \end{array}\right\} \neq \text{equal.}$$

inconsistant.

Hence we cannot write this point as a linear combination.

b) Find a basis for W and determine its independent-number (i.e., $dim(W)$)

From before.

$$\left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{array}\right] \Rightarrow \text{two leaders in 3 colums.}$$

each point written as a Column.

So for basis original → go to.

basis for $W = \{(-2, 1, 3), (2, 0, 5)\}$.

$\dim(W) = 2$.

Q7 **(10 pts)** Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $T((1,1,1)) = (4,1,3)$, $T((1,0,0)) = (2,1,0)$, and $T((0,0,1)) = (1,0,1)$
Find the standard matrix representation, M, of T.

We know the standard ordered basis of domain's transformations form columns for M.

Hence $T(1,0,0) = (2,1,0) \rightarrow$ as Col 1 of M

$T(0,0,1) = (1,0,1) \longrightarrow$ column 3 of M

$$T\big(\alpha_1(1,1,1) + \alpha_2(1,0,0) + \alpha_3(0,0,1)\big) =$$
$$\alpha_1 T(1,1,1) + \alpha_2 T(1,0,0) + \alpha_3 T(0,0,1)$$

Let $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = -1$

$$T\big((1,1,1) - (1,0,0) - (0,0,1)\big) = T(1,1,1) - T(1,0,0) - T(0,0,1)$$

$$T(0,1,0) = (4,1,3) - (2,1,0) - (1,0,1)$$
$$= (1,0,2) \longrightarrow \underline{\text{col } 2.}$$

Hence

$$M = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Check.

Point $(1,1,1)$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \text{correct.}$$

Q8 **(10 pts)** Let $T : R^{2\times 2} \longrightarrow \mathcal{P}_3$ be a function such that

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 2ax^2 + (b+c).$$

No x term.

1. Find the fake-function F that corresponds to the given function T. Use the fake-function F in order to answer the following questions about T.

$F : \mathbb{R}^4 \to \mathbb{R}^3 \quad T(a,b,c,d) = (2a, 0, b+c).$

a) Convince me that that T is a linear transformation.

Can we write the range as span of variables in domain.? In fake.

Fake Range = Span$\{(2,0,0), (0,0,1), \cancel{(0,0,1)}\}$.

written as span $\longrightarrow$ Yes

b) Find a basis for $Z(T)$, the zeros of T (i.e., N(T) or Ker(T)).

$\Rightarrow$ ~~To~~ Find a Matrix.

$$\text{Fake } M = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$Z(M) = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$a = 0 \quad b = -c$ and $d \in \mathbb{R}$.

Fake $Z(M) = \{(0, b, -b, d) \mid b \text{ and } d \in \mathbb{R}\}$

Fake $Z(M) = \text{span}\{(0,1,-1,0), (0,0,0,1)\}$

Basis for fake $= \{(0,1,-1,0), (0,0,0,1)\}$

Basis for Real $= \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

c) Determine whether T is onto or not.

T is onto when dim Range (M) = dim Codomain.

As before in a) Fake Range = Span$\{(2,0,0), (0,0,1)\}$

dim of Fake = Real = $\underline{\underline{2}}$

dim of $P_3 = 3$

not equal not $\underline{\underline{\text{Onto}}}$.

d) Is T an isomorphism? (brief lecture: T is called an isomorphism if it is one to one and onto). Show the work

Since it is not onto in c), not isomorphic.

not 1:1 Since $Z(M) \neq 0$ of $R^{2\times 2}$

Q9 **(10 pts)** Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

a) Find all eigenvalues of A. Then for each eigenvalue a of A find E_a.

$$|\alpha I_3 - A| = 0$$

$$\left|\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ +1 & -1 & 3 \end{bmatrix}\right| = 0$$

$$\left|\begin{matrix} \alpha-4 & 0 & 0 \\ -1 & \alpha-3 & +1 \\ -1 & 1 & \alpha-3 \end{matrix}\right| = 0.$$

$$C_\alpha(A) = (\alpha-4)(-1)^{1+1}\left[(\alpha-3)^2 - (1)\right] = 0$$

$$(\alpha-4)\left[\alpha^2 - 6\alpha + 9 - 1\right] = 0.$$

$$\alpha = 4, 4, 2$$

$$E_4 = Z[4I_3 - A]$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$X_1 = X_2 + X_3.$

$X_2, X_3 \in \mathbb{R}.$

$$E_4 = \text{Span}\{(1,1,0), (\cancel{0}1,0,1)\}$$

$$E_2 = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right. \xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$X_1 = 0$

$X_2 = X_3$

$$E_2 = \text{Span}\{(0,1,1)\}$$

b) Is A diagonalizable? If so, find an invertible matrix Q and a diagonal matrix D such that $QDQ^{-1} = A$

Yes diagonalizable, 4 repeated twice, span of 2 points, correct

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & \cancel{0} & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Q10 **(5 pts)** Find the value of α such that the points $\alpha u + v$ and w are orhogonal to each other in $\mathbb{R}^4$ where $u = (3, 0, -3, 0)$, $v = (-1, 4, 1, -2)$, $w = (4, -1, -4, -4)$

$$\alpha u + v = (3\alpha, 0, -3\alpha, 0) + (-1, 4, 1, -2).$$

$$\alpha u + v = (3\alpha - 1, 4, -3\alpha + 1, -2).$$

$$(\alpha u + v) \cdot w = 0$$

$$(3\alpha - 1, 4, -3\alpha + 1, -2) \cdot (4, -1, -4, -4) = 0.$$

$$4(3\alpha - 1) + 4 - 4(-3\alpha + 1) + 8 = 0.$$

$$12\alpha - 4 - 4 + 12\alpha - 4 + 8 = 0$$

$$24\alpha = -8 + 4 + 4 + 4.$$

$$\alpha = \frac{4}{24} = \frac{1}{6}.$$

$$\left(\frac{3}{6} - 1, 4, -\frac{3}{6} + 1, -2\right) \cdot (4, -1, -4, -4) = 0.$$

$$-\frac{1}{2} \times 4 - 4 + \frac{1}{2} \times -4 + 8 = 0.$$

$$0 = 0$$ correct.

$\boxed{Q11}$ **(10 pts)** Apply the Gram-Schmidt algorithm to the following points to obtain an orthogonal basis for $\mathbb{R}^3$. Do not reduce the given points to three different points. Apply the algorithm directly on Q_1, Q_2, Q_3.

$Q_1 = (1, 1, 1)$, $Q_2 = (0, 2, 2)$, and $Q_3 = (0, -2, 2)$.

$$\vec{W_1} = (1, 1, 1) \qquad |W_1|^2 = \underline{\underline{3}}.$$

$$\vec{W_2} = Q_2 - \frac{Q_2 \cdot W_1}{|W_1|^2} W_1 \qquad Q_2 \cdot W_1 = 4$$

$$W_2 = (0, 2, 2) - \frac{4}{3}(1, 1, 1) = \left(\frac{-4}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$W_3 = (0, -2, 2) - \cancel{\frac{Q_3 \cdot W_2}{|W_2|^2}} W_2 - \cancel{\frac{Q_3 \cdot W_1}{|W_1|^2}} W_1.$$

$$Q_3 \cdot W_2 = (0, -2, 2) \cdot \left(\frac{-4}{3}, \frac{2}{3}, \frac{2}{3}\right) = \underline{\underline{0}}$$

$$Q_3 \cdot W_1 = (0, -2, 2) \cdot (1, 1, 1) = \underline{\underline{0}}$$

$$W_3 = (0, -2, 2).$$

An orthogonal basis $= \left\{(1,1,1), \left(-4/3, \frac{2}{3}, \frac{2}{3}\right), (0, -2, 2)\right\}.$

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